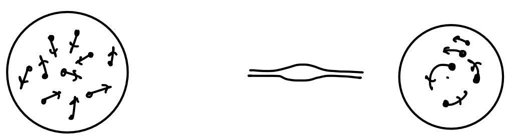
Kinetic Theory of Stellar Systems

WEEK 1

1. Introduction

Two mental models

Globular cluster — Disk galaxy



"hot sphere"
$N \sim 10^5, 10^6$
$R \sim 1\,\mathrm{pc} = 2\times10^5\,\mathrm{AU}$
$t_{dyn} \sim 10^5\,\mathrm{yr}$
$\mathrm{age} \sim 10^{10}\,\mathrm{yr}$

"cold axisymmetric disk"
$N \sim 10^{11}$
$R \sim 10\,\mathrm{kpc}$
$t_{dyn} \sim 10^8\,\mathrm{yr}$
$\mathrm{age} \sim 10^{10}\,\mathrm{yr}$

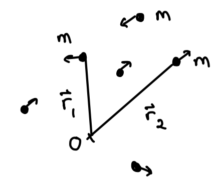
These systems are stable/weakly unstable (after "violent relaxation").
Key Q: how do such systems evolve over many Gyr?

Rough answer: hot sphere $\leftrightarrow$ gas (Dellar)
cold disk $\leftrightarrow$ plasma (Schekochihin)

BUT there are new behaviors unique to self-grav. systems.

Virial equilibrium

Assume equal masses m.



$$U(\vec{r}_1, \vec{r}_2) = -\frac{Gm^2}{|\vec{r}_1 - \vec{r}_2|}$$

Newton II:

$$m\frac{d\vec{v}_i}{dt} = -\frac{\partial}{\partial \vec{r}_i}\sum_{j\neq i} U(\vec{r}_i, \vec{r}_j)$$

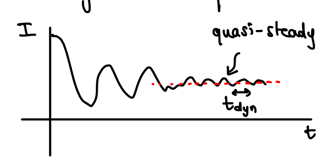
$$= -Gm^2 \sum_{j\neq i} \frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^3}$$

Now consider moment of inertia

$$I \equiv \sum_i m\vec{r}_i^{\,2}$$

$$\Rightarrow \frac{dI}{dt} = 2\sum_i m\vec{r}_i\cdot\vec{v}_i$$

e.g. cold collapse



$$\Rightarrow \frac{1}{2}\frac{d^2 I}{dt^2} = \sum_i m\vec{v}_i^{\,2} + \sum_i m\vec{r}_i\cdot\frac{d\vec{v}_i}{dt}$$

$$= \sum_i m\vec{v}_i^{\,2} - \sum_i m\vec{r}_i\cdot Gm\sum_{j\neq i}\frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^3}$$

$$= \sum_i m\vec{v}_i^{\,2} - \frac{1}{2} G m^2 \sum_i \sum_{j \neq i} \frac{(\vec{r}_i - \vec{r}_j)\cdot(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^3}$$

$$= \sum_i m\vec{v}_i^{\,2} + \sum_i \sum_{j>i} U(\vec{r}_i, \vec{r}_j).$$

Now avg. $\langle \cdots \rangle$ over several t_{dyn}:

$$\frac{1}{2} \frac{d^2 \langle I \rangle}{dt^2} = 0$$

$$\Rightarrow \underbrace{\left\langle \sum_i m\vec{v}_i^{\,2} \right\rangle}_{\equiv 2K} + \underbrace{\left\langle \sum_i \sum_{j>i} U(\vec{r}_i, \vec{r}_j) \right\rangle}_{\equiv W} = 0$$

$$\Rightarrow \boxed{2K + W = 0} \qquad \text{Virial theorem.}$$

→ roughly, KE and PE are balanced in near-eqm systems.

Allows us to estimate things like velocity dispersion in hot spheres:

$$K \sim \frac{1}{2} M \sigma^2, \quad W \sim -\frac{GM}{R}$$

$$\Rightarrow \sigma \sim \sqrt{\frac{GM}{R}} \sim 10\,\text{km/s} \left(\frac{M}{10^5 M_\odot}\right)^{1/2} \left(\frac{R}{\text{pc}}\right)^{-1/2}$$

<u>Thermal equilibrium</u>?

How would we construct thermodynamics for stellar system? Define temperature:

$$K = \frac{3}{2} N k_B T$$

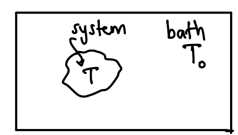
From virial theorem,

$$E_{tot} = K + W = K + (-2K) = -K$$

$$= -\frac{3}{2} N k_B T < 0 \qquad (\text{bound})$$

$$\Rightarrow C \equiv \frac{\partial E_{tot}}{\partial T} = -\frac{3}{2} N k_B < 0 \qquad (!!!)$$

This implies thermal equilibrium impossible. Why? Suppose system sat in heat bath at T_0:



initially $T = T_0$.

Let system lose energy $|\Delta E|$. (negative fluctuation)

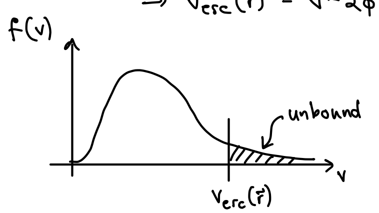
$$\Rightarrow \delta T = \frac{\delta E}{C} = \frac{-|\Delta E|}{-\frac{3}{2} N k_B} = \frac{2|\Delta E|}{3 N k_B} > 0$$

→ system gets hotter
→ loses more energy!

What about an isolated system?

No Maxwellian: $E = \frac{v^2}{2} + \phi(\vec{r})$

$$\Rightarrow v_{esc}(\vec{r}) = \sqrt{-2\phi(\vec{r})}$$



("evaporation")

What is the shaded fraction? It's $\vec{r}$-dependent:

$$f_{unbound} = \frac{\int_{v_{esc}(\vec{r})}^{\infty} dv \cdot e^{-3v^2/2\sigma^2}}{\int_0^{\infty} dv \cdot e^{-3v^2/2\sigma^2}}$$

To get an estimate, replace $v_{esc}(\vec{r})$ with its mass-weighted avg:

$$\sigma_{esc}^2 \sim \frac{1}{M}\int d\vec{r}\, \rho(\vec{r})\, v_{esc}^2(\vec{r})$$

$$= -\frac{2}{M}\int d\vec{r}\, \rho(\vec{r})\, \phi(\vec{r})$$

$$= -\frac{4}{M} \cdot \underbrace{\frac{1}{2}\int d\vec{r}\, \rho(\vec{r})\, \phi(\vec{r})}_{= W}$$

$$= -\frac{4W}{M} = \frac{8K}{M} \quad \text{by virial.}$$

Then $K \sim \frac{1}{2} M\sigma^2 \Rightarrow \sigma_{esc}^2 \sim 4\sigma^2$

$$\Rightarrow \sigma_{esc} \sim 2\sigma$$

$$\Rightarrow f_{unbound} \sim 1\% .$$

Key point: evaporation is not a particularly rare process, way out in the tail of the DF; it's common!

Replenish the tail every $\sim t_{relax}$ $\Rightarrow$ clusters can lose signif. fraction of M in lifetime!

<u>What next?</u>

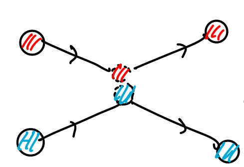
Virial eqm but no thermal eqm.

→ stellar systems really in quasi-eqm on time $\sim t_{relax}$

time to "forget ICs" (<u>not</u> time to reach Maxw., like in gas/plasma).

Next: simplest estimate of t_{relax}.

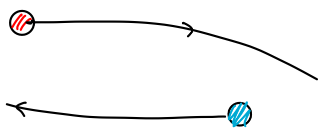
Gas:



$\Delta v \sim v_0 \sim v_{th}$

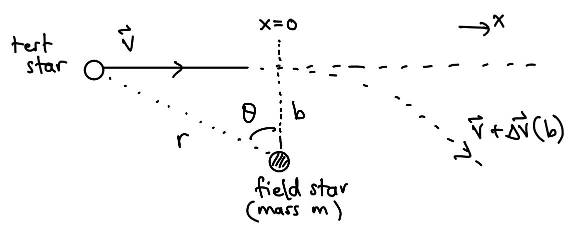
$$\Rightarrow t_{relax} \sim \frac{\lambda_{mfp}}{v_{th}}$$

Plasma / galaxy :



$\Delta v \ll \sigma$

(Spitzer/) Chandrasekhar theory of 2-body relaxation



Weak encounter $\Delta v \ll v$. Zeroth order orbit : straight line.
Next order :

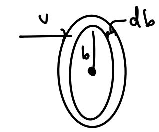
$\rightarrow$ symmetry , $\Delta v_{\parallel} = 0$

$\rightarrow$ $$\Delta v_{\perp} = \int dt \cdot a_{\perp}(t)$$
$$= \int dt \, \frac{Gm}{r} \cos\theta$$
$$= \int_{-\infty}^{\infty} \frac{dx}{v} \, \frac{Gm\,b}{(r^2+x^2)^{3/2}}$$
$$= \frac{2Gm}{bv}$$

$x = vt$
$r = \sqrt{x^2+b^2}$
$\cos\theta = \frac{b}{r}$

How many nudges until $\langle(\vec{\Delta v})^2\rangle \sim v^2$? Well, in time T,

$$\langle(\vec{\Delta v})^2\rangle_T \approx \int db \cdot 2\pi b \cdot n v T \times \left(\frac{2Gm}{bv}\right)^2$$



$$= \frac{8\pi G^2 m^2 n T}{v} \underbrace{\int \frac{db}{b}}_{= \ln \frac{b_{max}}{b_{min}} \equiv \ln \Lambda}$$

Coulomb logarithm.

(Usually take $b_{max} \sim R$, $b_{min} \sim b_{90}$).

Relaxation time : set $\langle(\vec{\Delta v})^2\rangle_T = v^2$:

$$T = \frac{v^3}{8\pi G^2 m^2 n \ln \Lambda} \equiv t_{2BR}$$ two body relaxation time

Let's express $T = T(N, t_{dyn})$ using $v^2 \sim \frac{Gmn}{R}$,
$v \sim t_{dyn}^{-1} R$
$n \sim \left(\frac{4\pi R^3}{3}\right)^{-1} N$

$$\Rightarrow t_{2BR} \sim \frac{0.1\, N t_{dyn}}{\ln N}$$
$$\sim N t_{dyn} = \begin{cases} \sim 10 \text{ Gyr}, & \text{glob clust.} \\ \sim 10^9 \text{ Gyr}, \gg t_{universe} & \text{disk gal.} \end{cases}$$

For glob. clusters, 2BR is main physical mechanism driving long-term evolution ("relaxation"). But this is clearly not so for disk galaxies.

What's gone wrong?

Chandra's theory has several shortcomings when applied to galactic disks:

- inhomogeneity
- orbits $\neq$ straight lines
- repeated interactions ($\Rightarrow$ resonances)
- collective effects

2. Orbits ("how to build a galaxy")

On timescales $\ll t_{relax}$, care only about mean-field motion.

(Analogies: · homogeneous plasma $\vec{x} = \vec{v}_0 t + \vec{x}_0$;
· solar system: Keplerian ellipse around Sun).

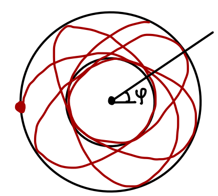
Introduce $\phi_0(\vec{r})$ "mean-field potential". (We'll assess the effect of fluctuations $\delta\phi(\vec{r},t)$ later).

Eqns of motion of test star:

$$\frac{d\vec{r}}{dt} = \vec{v}, \qquad \frac{d\vec{v}}{dt} = -\frac{\partial\phi}{\partial\vec{r}}$$

In general, can't solve analytically, but trivial numerically. For our purposes, 1 orbit family: rosettes (in central potentials $\phi = \phi(R)$).

(φ, R)



Orbit labels (r_p, r_a) peri/apocentre

superp. of 2 oscill!

Central potentials: planar orbits. Constants of motion

$$L = R^2 \frac{d\varphi}{dt}, \qquad E = \frac{1}{2}\left(\frac{dR}{dt}\right)^2 + \frac{L^2}{2R^2} + \phi(R)$$

alternative orbit labels

$$v_R(r_{p/a}) = 0 \quad \Rightarrow \quad \sqrt{2(E - \phi(r_{p/a})) - L^2/r_{p/a}^2} = 0$$

Exercise: show

$$E(r_p, r_a) = \frac{r_a^2 \phi(r_a) - r_p^2 \phi(r_p)}{r_a^2 - r_p^2}$$

$$L(r_p, r_a) = \sqrt{\frac{2[\phi(r_p) - \phi(r_a)]}{r_p^{-2} - r_a^{-2}}}$$

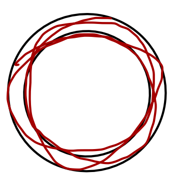
Can also get frequencies. e.g.

$$\Omega_R = \frac{2\pi}{T_R},$$

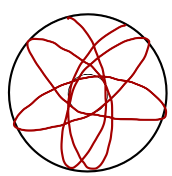
$$T_R(r_p, r_a) = \int \frac{dR}{v_R} = 2\int_{r_p}^{r_a} \frac{dR}{\sqrt{2(E(r_p,r_a) - \phi(R)) - L^2(r_p,r_a)/R^2}}$$

...similarly for T_ϕ, Ω_ϕ, $\Delta\phi$, etc.

Key idea: 2 numbers label the orbit. (In 3D, 3 numbers!).



$r_p \approx r_a$



$r_p \ll r_a$

All we don't have is phases; quasi-eqm. is collection of orbits with random phases.

Phases: quasiperiodicity

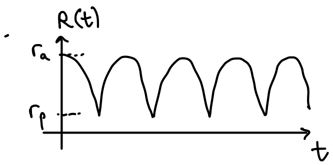
Numerical orbit integration:

$$\vec{r}(t) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{\Omega}t}\, \vec{r}_k$$

$(\Omega_1, \Omega_2, \Omega_3)$ characteristic frequencies

integers, e.g. $(0,1,3)$, $(-4,1,7)$ etc.

e.g.



dominated by small # of $\vec{k}$ values.

$\Rightarrow$ orbit is superposition of oscillators. This suggests we use a new coordinate system...

Angle-action variables

Canonical coordinates $(\vec{q}, \vec{p})$ (e.g. $\vec{x}, \vec{v}$). Hamilton's eqns:

$$\frac{d\vec{q}}{dt} = \frac{\partial H}{\partial \vec{p}}, \qquad \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{q}}$$

Poisson bracket

$$[g, h] = \frac{\partial g}{\partial \vec{q}} \cdot \frac{\partial h}{\partial \vec{p}} - \frac{\partial g}{\partial \vec{p}} \cdot \frac{\partial h}{\partial \vec{q}}$$

What is a good choice of $(\vec{q}, \vec{p})$? In plasma, $(\vec{x}, \vec{v})$. Why? Because $\vec{x} = \vec{v}t + \vec{x}_0$ and $\vec{v}$ is conserved. Key idea: it's good to use conserved quantities as canonical momenta!

In general, an integral of motion $I(\vec{x},\vec{v})$ is any function conserved along an orbit. We'd like to choose $\vec{p} = \vec{I}$, but will this be canonical (i.e. preserve Poisson bracket & Hamilton's eqns)?

It turns out <u>yes, but</u> only if we use a special choice of I: <u>actions</u>:

$$J_i \equiv \oint_{\gamma_i} \frac{1}{2\pi} \vec{p} \cdot d\vec{q}$$

$\{\gamma_i\}$ are 3 topologically-distinct paths through orbital phase space.

Taking $\vec{p} = \vec{J}$, $\exists$ complimentary canonical coords "<u>angles</u>" $\vec{\theta}$:

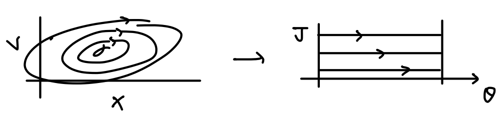
$$H = H(\vec{J})$$

$$\frac{d\vec{J}}{dt} = \frac{\partial H}{\partial \vec{\theta}} = 0 \quad \Rightarrow \quad \vec{J} = \vec{J}_0 = \text{const}.$$

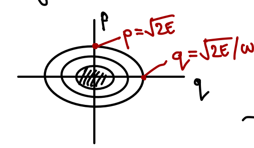
$$\frac{d\vec{\theta}}{dt} = \frac{\partial H}{\partial \vec{J}} \equiv \vec{\Omega}(\vec{J}) = \text{const}$$

$$\Rightarrow \vec{\theta} = \vec{\theta}_0 + \vec{\Omega} t \mod 2\pi.$$

$$[g,h] = \frac{\partial g}{\partial \vec{\theta}} \cdot \frac{\partial h}{\partial \vec{J}} - \frac{\partial g}{\partial \vec{J}} \cdot \frac{\partial h}{\partial \vec{\theta}}$$



<u>E.g. harmonic oscillator</u> $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$.



$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q$$

$$\frac{dq}{dt} = p$$

$$J = \frac{1}{2\pi}\int p \cdot dq = \frac{\text{(area)}}{2\pi}$$

$$= \frac{\pi \frac{\sqrt{2E}}{\omega} \cdot \sqrt{2E}}{2\pi}$$

$$= \frac{E}{\omega}$$

$$\Rightarrow H = \omega J, \quad \Omega = \omega$$

$$\theta = \omega t + \theta_0 \mod 2\pi$$

Can show (type-2 canonical transform). $p = \omega^2 q \cot\theta$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\omega^2 q}{p}\right)$$

<u>E.g. central orbits</u>

$\vec{J} = (J_1, J_2) = (L, J_R)$, $\quad \vec{\theta} = (\theta_1, \theta_2)$

- $L = \frac{1}{2\pi}\int d\varphi\, p_\varphi = R v_\varphi$
- $J_R = \frac{1}{2\pi}\int dR\, p_R = \frac{1}{\pi}\int_{r_p}^{r_a} dR \sqrt{2(E-\phi(R)) - L^2/R^2}$
- $\theta_1 = \int_{r_p}^{R} dR' \frac{\Omega_\varphi - L^2/R'^2}{v_R(R')}$, $\quad \theta_2 = \Omega_R \int_{r_p}^{R} \frac{dR'}{v_R(R')}$

→ Much of modern Galactic astronomy (e.g. GAIA) now done in AA variables.

→ Natural variables for kinetic theory.

→ Near equilibrium, $f(\vec{r},\vec{v},t) \approx f(\vec{J})$ (Integrals conserved; no $\vec{\theta}$ or t-dependence).

→ Also tokamaks, magnetospheres.

From now, always assume $(\vec{\theta},\vec{J})$ exist.

Roughly, $(\vec{x},\vec{v})_{\text{plasma}} \longleftrightarrow (\vec{\theta},\vec{J})_{\text{galaxy}}$. But 2 major complications:

- $\vec{\Omega} = \vec{\Omega}(\vec{J})$ nonlinear.
- $\phi = \phi(\vec{\theta},\vec{J})$ depends on momenta, not just coords.

3. Fundamentals of kinetic theory

Klimontovich DF

$$f^M(\vec{r},\vec{v},t) = m\sum_{i=1}^{N} \delta(\vec{r}-\vec{r}_i(t))\,\delta(\vec{v}-\vec{v}_i(t))$$

Normalized s.t. $\int d\vec{r}\,d\vec{v}\,f^M = \int d\vec{r}\,\rho^M = Nm = M$. Thus

$$\phi^M(\vec{r},t) = \int d\vec{r}'\,d\vec{v}'\,\psi(\vec{r},\vec{r}')\,f^M(\vec{r}',\vec{v}',t)$$

where $\psi(\vec{r},\vec{r}') = \dfrac{-G}{|\vec{r}-\vec{r}'|}$. (This is just the soln to $\nabla^2\phi^M = 4\pi G\rho^M$).

Eqns of motion:

$$H^M(\vec{r},\vec{v},t) = \frac{\vec{v}^2}{2} + \phi^M(\vec{r},t)$$

$$\frac{d\vec{r}_i}{dt} = \frac{\partial}{\partial\vec{v}_i}H^M(\vec{r}_i,\vec{v}_i,t)$$

$$\frac{d\vec{v}_i}{dt} = -\frac{\partial}{\partial\vec{r}_i}H^M(\vec{r}_i,\vec{v}_i,t) = -\frac{\partial}{\partial\vec{r}_i}\phi(\vec{r}_i,t)$$

Equivalent to Klimontovich equation

$$\frac{df^M}{dt} = \frac{\partial f^M}{\partial t} + \vec{v}\,\frac{\partial f^M}{\partial\vec{r}} - \frac{\partial\phi^M}{\partial\vec{r}}\cdot\frac{\partial f^M}{\partial\vec{v}} = 0.$$

Coordinate-free:

$$\frac{df^M}{dt} = \frac{\partial f^M}{\partial t} + [f^M, H^M] = 0. \qquad \text{Liouville!}$$

For later use:

$$f^M(\vec{\theta},\vec{J},t) = m\sum_{i=1}^{N} \delta(\vec{\theta}-\vec{\theta}_i(t))\,\delta(\vec{J}-\vec{J}_i(t))$$

$$\phi^M(\vec{\theta},\vec{J},t) = \int d\vec{\theta}'\,d\vec{J}'\;\psi(\vec{\theta},\vec{J},\vec{\theta}',\vec{J}')\,f^M(\vec{\theta}',\vec{J}',t).$$

<u>Mean-field + fluctuations</u>

W.l.o.g., write:

$$f^M = f_0 + \delta f$$
$$\phi^M = \phi_0 + \delta\phi$$
$$H^M = H_0 + \delta\phi$$

where $\langle \delta f\rangle = 0$ etc, under some averaging procedure. Then

$$\frac{\partial f_0}{\partial t} + \frac{\partial \delta f}{\partial t} + [f_0, H_0] + [\delta f, H_0] + [f_0, \delta\phi] + [\delta f, \delta\phi] = 0.$$

Averaging this,

$$\frac{\partial f_0}{\partial t} + [f_0, H_0] = -\left\langle [\delta f, \delta\phi]\right\rangle \qquad (1)$$

Subtract:

$$\frac{\partial \delta f}{\partial t} + [\delta f, H_0] + [f_0, \delta\phi] = -\Big([\delta f, \delta\phi] \quad \left\langle [\delta f, \delta\phi]\right\rangle \Big) \qquad (2)$$

We'll take $\langle \ldots \rangle = \frac{1}{(2\pi)^d}\int d\vec{\theta}\,(\ldots)$ <u>angle average</u>.

In the case of pure axisymmetry ($\vec{\theta}$-independent), (1) implies

$$\frac{\partial f_0}{\partial t} + [f_0, H_0] = 0$$

Thus any f_0 that "commutes" with H_0 is time-independent.
→ "collisionless equilibrium".

→ Kinetic theory answers the Q: how does f_0 evolve (slowly) from one collisionless eqm to another, in the presence of (non-axisymm, $\vec{\theta}$-dep.) fluctuations (e.g. due to finite N)?

Linear theory. Assume $\delta\phi$, δf <u>small</u>. Then (2) gives

$$\frac{\partial \delta f}{\partial t} + [\delta f, H_0] + [f_0, \delta\phi] \simeq 0. \qquad (3)$$

The rest of the course will focus on (1) and (3)
(and $\nabla^2\delta\phi = 4\pi G\,\delta\rho$ if we do self-consistent problems).

We'll assume δf, $\delta\phi$ change on $\sim t_{dyn}$
" " f_0, H_0 " " $\sim t_{relax} \gg t_{dyn}$

→ can solve for δf, $\delta\phi$ while keeping f_0, H_0 fixed.

__A-A variables__

$$f_0 = \frac{1}{(2\pi)^d}\int d\vec\theta\, f^M$$

$$H_0 = H_0(\vec J) \quad \text{by construction of the AA variables.}$$

$$\Rightarrow \ [f_0, H_0] = 0$$

$$\Rightarrow \ \frac{\partial f_0}{\partial t} = 0 \quad \text{for any } f_0(\vec J)$$

Jeans' theorem: any function of actions is coll. eqm.

Next, express

$$f(\vec\theta, \vec J, t) = f_0(\vec J) + \delta f(\vec\theta, \vec J, t)$$

$$H(\vec\theta, \vec J, t) = H_0(\vec J) + \delta\phi(\vec\theta, \vec J, t)$$

Let's rewrite (1) and (3) in the AA variables. First, (1):

$$\frac{\partial f_0}{\partial t} = -\langle[\delta f, \delta\phi]\rangle$$

$$= -\frac{1}{(2\pi)^d}\int d\vec\theta\left(\frac{\partial\delta f}{\partial\vec\theta}\cdot\frac{\partial\delta\phi}{\partial\vec J} - \frac{\partial\delta f}{\partial\vec J}\cdot\frac{\partial\delta\phi}{\partial\vec\theta}\right)$$

$$= -\frac{1}{(2\pi)^d}\int d\vec\theta\left(\frac{\partial}{\partial\vec\theta}\cdot\left(\delta f\,\frac{\partial\delta\phi}{\partial\vec J}\right) + \frac{\partial}{\partial\vec J}\cdot\left(-\delta f\cdot\frac{\partial\delta\phi}{\partial\vec\theta}\right)\right)$$

$$= +\frac{\partial}{\partial\vec J}\cdot\int\frac{d\vec\theta}{(2\pi)^d}\,\delta f\,\frac{\partial\delta\phi}{\partial\vec\theta} \qquad \text{(totally general!)} \qquad (4)$$

Next, (3):

$$\frac{\partial\delta f}{\partial t} + \vec\Omega\cdot\frac{\partial\delta f}{\partial\vec\theta} - \frac{\partial f_0}{\partial\vec J}\cdot\frac{\partial\delta\phi}{\partial\vec\theta} = 0. \qquad (5)$$

[Similarly, for later use,

$$\delta\phi(\vec\theta, \vec J, t) = \int d\vec\theta'\, d\vec J'\, \psi(\vec\theta, \vec J, \vec\theta', \vec J')\, \delta f(\vec\theta', \vec J', t)$$]

Finally, it'll be helpful to write (4)-(5) in Fourier space:

$$\delta f(\vec\theta, \vec J, t) = \sum_{\vec k} e^{i\vec k\cdot\vec\theta}\, \delta f_{\vec k}(\vec J, t) \quad \text{etc.}$$

$$\Rightarrow (4)_{\vec k}: \quad \frac{\partial f_0}{\partial t} = -i\frac{\partial}{\partial\vec J}\cdot\sum_{\vec k}\vec k\,\delta\phi_{\vec k}^*\,\delta f_{\vec k}$$

$$(5)_{\vec k}: \quad \frac{\partial\delta f}{\partial t} + i\vec k\cdot\vec\Omega\,\delta f_{\vec k} - i\vec k\cdot\frac{\partial f_0}{\partial\vec J}\,\delta\phi_{\vec k} = 0.$$

[Similarly,

$$\delta\phi_{\vec k}(\vec J, t) = (2\pi)^3\sum_{\vec k'}\int d\vec J'\, \psi_{\vec k\vec k'}(\vec J, \vec J')\, \delta f_{\vec k'}(\vec J')$$

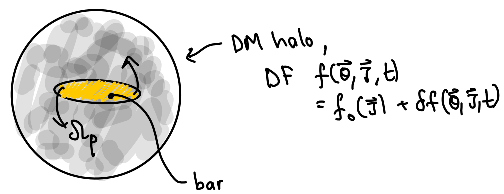
= Poisson's eqn in Fourier space!

... in plasma, $\psi_{\vec k\vec k'} \propto \delta_{\vec k}^{\vec k'}\cdot\frac{1}{k^2}$, easy!

Here, depends on mapping $\vec r(\vec\theta, \vec J)$. Not easy!] -

End of day 1 (t = 3.5 hours).

4. <u>Example: Torque on a galactic bar.</u>



$$H = \frac{v^2}{2} + \underbrace{\phi(r)}_{\text{central}} + \underbrace{\delta\phi(\vec r,t)}_{\text{due to bar (no self-cons.)}}$$

AA variables are $(\theta_\varphi, \theta_\psi, \theta_R)$, (L_z, L, J_R).
Torque on 1 DM halo particle:

$$\frac{dL_z}{dt} = -\frac{\partial}{\partial\theta_\varphi} H$$

$$= -\frac{\partial\delta\phi}{\partial\theta_\varphi}$$

$$\Rightarrow \text{torque on bar} = -\int d\vec\theta\, d\vec J\, f(\vec\theta,\vec J,t)\,\frac{dL_z}{dt}$$

$$= \int d\vec\theta\, d\vec J\, \delta f(\vec\theta,\vec J,t)\,\frac{\partial\delta\phi(\vec\theta,\vec J,t)}{\partial\theta_\varphi}$$

$$= \int d\vec\theta\, d\vec J \sum_{\vec k} e^{i\vec k\cdot\vec\theta}\,\delta f_{\vec k} \sum_{\vec k'} i k'_\varphi\,\delta\phi_{\vec k'}\, e^{i\vec k'\cdot\vec\theta}$$

$$= \sum_{\vec k\vec k'} \int d\vec J\, \delta f_{\vec k}\,\delta\phi_{\vec k'} \underbrace{\int d\vec\theta\, e^{i(\vec k+\vec k')\cdot\vec\theta}}_{=(2\pi)^d\delta_{\vec k}^{-\vec k'}} i k'_\varphi$$

$$= \sum_{\vec k} \int d\vec J\, \delta f_{\vec k}\,\delta\phi^*_{\vec k}\,(2\pi)^d\, i(-k_\varphi)$$

$$= -i(2\pi)^d \sum_{\vec k} k_\varphi \int d\vec J\, \delta f_{\vec k}(\vec J,t)\,\delta\phi^*_{\vec k}(\vec J,t) \qquad (*).$$

One can show that $\delta\phi = A(R)\cos(2(\varphi-\Omega_p t))$ becomes, after FT in AA variables, of the form,

$$\delta\phi(\vec r,t) = \sum_{\vec k} \Phi_{\vec k}(\vec J)\, e^{i\vec k\cdot\vec\theta}\, e^{-ik_\varphi\Omega_p t}$$

To see this, use $\varphi = \theta_\varphi + \lambda(\theta_R,\vec J)$ and calculate $\delta\phi_{\vec k}(\vec J,t) = \int \frac{d\vec\theta}{(2\pi)^d} e^{-i\vec k\cdot\vec\theta} A\cos(..)$ $\propto e^{-ik_\varphi\Omega_p t}\left(\delta_m^{k_\varphi} + \delta_m^{-k_\varphi}\right)$

And the solution to $(5)_{\vec k}$ is

$$\delta f_{\vec k}(\vec J,t) = \delta f_{\vec k}(\vec J,0)\, e^{-i\vec k\cdot\vec\Omega t} + \int_0^t dt'\, e^{-i\vec k\cdot\vec\Omega(t-t')}\,\delta\phi_{\vec k}(\vec J,t)\, i\vec k\cdot\frac{\partial f_0}{\partial\vec J}$$

Hence, for $\delta f_{\vec k}(t=0) = 0$

$$T(t) = -i(2\pi)^d \sum_{\vec k} k_\varphi \int d\vec J\, i\vec k\cdot\frac{\partial f_0}{\partial\vec J} \int_0^t dt'\, e^{-i\vec k\cdot\vec\Omega(t-t')}\,\delta\phi_{\vec k}(\vec J,t')\,\delta\phi^*_{\vec k}(\vec J,t)$$

$$= (2\pi)^d \sum_{\vec k} \vec k k_\varphi \int d\vec J\, \frac{\partial f_0}{\partial\vec J}\, |\Phi_{\vec k}(\vec J)|^2 \int_0^t dt'\, e^{-i\vec k\cdot\vec\Omega(t-t')}\, e^{-ik_\varphi\Omega_p t'}\, e^{ik_\varphi\Omega_p t}$$

$$= (2\pi)^d \sum_{\vec k} \vec k k_\varphi \int d\vec J\, \frac{\partial f_0}{\partial\vec J}\, |\Phi_{\vec k}(\vec J)|^2\, e^{-i(\vec k\cdot\vec\Omega - k_\varphi\Omega_p)t} \underbrace{\int_0^t dt'\, e^{i(\vec k\cdot\vec\Omega-k_\varphi\Omega_p)t'}}_{=\frac{e^{i(\vec k\cdot\vec\Omega-k_\varphi\Omega_p)t}-1}{i(\vec k\cdot\vec\Omega-k_\varphi\Omega_p)}}$$

$$= (2\pi)^d \sum_{\vec k} \vec k k_\varphi \cdot \int d\vec J \cdot \frac{\partial f_0}{\partial\vec J}\, |\Phi_{\vec k}(\vec J)|^2 \underbrace{\left\{\frac{1-e^{-i(\vec k\cdot\vec\Omega-k_\varphi\Omega_p)t}}{i(\vec k\cdot\vec\Omega - k_\varphi\Omega_p)}\right\}}_{\text{Real part } \to\, \pi\delta(..) \text{ as } t\to\infty.}$$

$$\Rightarrow \quad T(t\to\infty) = (2\pi)^d \sum_{\vec{k}} k_\varphi \vec{k}\cdot \int d\vec{J}\, \frac{\partial f_0}{\partial \vec{J}} |\Phi_{\vec{k}}(\vec{J})|^2 \pi \delta(\vec{k}\cdot\vec{\Omega} - k_\varphi \Omega_p)$$

For isotropic DFs, $f_0(\vec{r},\vec{v}) = f_0(r,v)$, it can be shown that $f_0(\vec{J}) = f_0(H_0(\vec{J})) = f_0(E)$. Then

$$\vec{k}\cdot\frac{\partial f_0}{\partial \vec{J}} = \vec{k}\cdot\frac{\partial H_0}{\partial \vec{J}}\frac{df_0}{dE} = \vec{k}\cdot\vec{\Omega}\,\frac{df_0}{dE}$$

$$\Rightarrow \delta(\vec{k}\cdot\vec{\Omega} - k_\varphi\Omega_p)\,\vec{k}\cdot\frac{\partial f_0}{\partial \vec{J}} = k_\varphi \Omega_p\, \delta(\vec{k}\cdot\vec{\Omega} - k_\varphi\Omega_p)\frac{df_0}{dE}$$

$$\Rightarrow T(t\to\infty) = (2\pi)^d \sum_{\vec{k}} k_\varphi^2 \int d\vec{J}\, |\phi_{\vec{k}}(\vec{J})|^2 \pi\, \delta(\vec{k}\cdot\vec{\Omega} - k_\varphi\Omega_p)\,\Omega_p \frac{df_0}{dE}$$

$$\Rightarrow \quad \mathrm{sgn}\left(T(t\to\infty)\right) = \mathrm{sgn}\left(\frac{df_0}{dE}\right)$$

All stable isotropic DFs $f_0(E)$ have $\frac{df_0}{dE} < 0$ (Antonov)

$\Rightarrow$ $T(t\to\infty) < 0$

$\Rightarrow$ torque on bar is negative

$\Rightarrow$ bar slows down!

End of day 2 ($t = 4.5$ hours).

WEEK 2

So far:

- stellar system = collection of orbits $(\vec{\theta},\vec{J})$
- mean field $H_0(\vec{J})$, $f_0(\vec{J})$, $\vec{\Omega}(\vec{J}) \equiv \partial H_0/\partial \vec{J}$
- nonaxisymmetric "fluctuations" $\delta f(\vec{\theta},\vec{J},t)$, $\delta\phi(\vec{\theta},\vec{J},t)$ $\sim \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{\theta}}\,\delta f_{\vec{k}}$ etc.
- fundamental eqn $\frac{df}{dt} = 0$
- axisymmetric part: $\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \vec{J}}\cdot \sum_{\vec{k}} i\vec{k}\,\delta\phi_{\vec{k}}^*\,\delta f_{\vec{k}}$
- linearized fluctuations satisfy

$$\frac{\partial}{\partial t}\delta f_{\vec{k}} + i\vec{k}\cdot\vec{\Omega}\,\delta f_{\vec{k}} - i\vec{k}\cdot\frac{\partial f_0}{\partial \vec{J}}\,\delta\phi_{\vec{k}} = 0. \qquad (1)$$

Solution is:

$$\delta f_{\vec{k}}(\vec{J},t) = \delta f_{\vec{k}}(\vec{J},0)\,e^{-i\vec{k}\cdot\vec{\Omega}t} + i\vec{k}\cdot\frac{\partial f_0}{\partial \vec{J}}\int_0^t dt'\, e^{-i\vec{k}\cdot\vec{\Omega}(t-t')}\,\delta\phi_{\vec{k}}(\vec{J},t')$$

- self-consistent problems also require

$$\delta\phi_{\vec{k}}(\vec{J},t) = (2\pi)^3 \sum_{\vec{k}'} \int d\vec{J}'\, \underbrace{\psi_{\vec{k}\vec{k}'}(\vec{J},\vec{J}')}_{\text{FT of } \psi = \frac{-G}{|\vec{r}-\vec{r}'|}}\, \delta f_{\vec{k}'}(\vec{J}',t) \qquad (2)$$

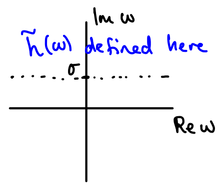
("bare coupling")

5. Self-consistent linear response theory

Eqns (1), (2) are an IVP. Use Laplace:

$$\tilde{h}(\omega) \equiv \int_0^t dt\, e^{i\omega t}\, h(t)$$

$$h(t) = \frac{1}{2\pi}\int_{i\sigma-\infty}^{i\sigma+\infty} e^{-i\omega t}\,\tilde{h}(\omega)$$



Laplace on (1):

$$-\delta f_{\vec{k}}(\vec{J},0) - i(\omega - \vec{k}\cdot\vec{\Omega})\,\widetilde{\delta f}_{\vec{k}}(\vec{J},\omega) = i\vec{k}\cdot\frac{\partial f_0}{\partial \vec{J}}\,\widetilde{\delta\phi}_{\vec{k}}(\vec{J},\omega)$$

$$\Rightarrow \widetilde{\delta f}_{\vec{k}}(\vec{J},\omega) = \frac{-i\vec{k}\cdot\frac{\partial f_0}{\partial \vec{J}}\,\widetilde{\delta\phi}_{\vec{k}}(\vec{J},\omega)}{i(\omega-\vec{k}\cdot\vec{\Omega})} - \frac{\delta f_{\vec{k}}(\vec{J},0)}{i(\omega-\vec{k}\cdot\vec{\Omega})}$$

Similarly for (2):

$$\widetilde{\delta\phi}_{\vec{k}}(\vec{J},\omega) = (2\pi)^3\sum_{\vec{k}'}\int d\vec{J}'\,\psi_{\vec{k}\vec{k}'}(\vec{J},\vec{J}')\,\widetilde{\delta f}_{\vec{k}'}(\vec{J}',\omega)$$

Eliminate $\widetilde{\delta f}$:

$$\widetilde{\delta\phi}_{\vec{k}}(\vec{J},\omega) = -(2\pi)^3\sum_{\vec{k}'}\int d\vec{J}'\,\frac{\psi_{\vec{k}\vec{k}'}(\vec{J},\vec{J}')\,\vec{k}'\cdot\frac{\partial f_0}{\partial \vec{J}'}\,\widetilde{\delta\phi}_{\vec{k}'}(\vec{J}',\omega)}{\omega-\vec{k}'\cdot\vec{\Omega}'}$$

$$-(2\pi)^3\sum_{\vec{k}'}\int d\vec{J}'\,\psi_{\vec{k}\vec{k}'}(\vec{J},\vec{J}')\,\frac{\delta f_{\vec{k}'}(\vec{J}',0)}{i(\omega-\vec{k}'\cdot\vec{\Omega}')} \qquad (*)$$

"Operator" notation:

$$|\delta\phi(\omega)\rangle = \hat{M}(\omega)|\delta\phi(\omega)\rangle + |\delta\phi^{\text{source}}(\omega)\rangle$$

$$\Rightarrow (\mathbb{1}-\hat{M}(\omega))|\delta\phi(\omega)\rangle = |\delta\phi^{\text{source}}(\omega)\rangle$$

- ω s.t. $\det(\mathbb{1}-\hat{M}(\omega)) \neq 0$,

  $$|\delta\phi(\omega)\rangle = (\mathbb{1}-\hat{M}(\omega))^{-1}|\delta\phi^{\text{source}}(\omega)\rangle$$

  "dressed noise"

  in plasma, $\delta\phi_k \sim \frac{1}{\epsilon_k}\,\delta\phi_k^{\text{source}}$

  true for most ω

- ω s.t. $\det(\mathbb{1}-M(\omega)) = 0$ (discrete set ω_g)

  $|\delta\phi(\omega)\rangle \sim e^{-i\omega_g t}|\psi_g\rangle$ Landau modes

→ In plasma course, you kept Landau modes, ignored noise (justified because the plasma was unstable!)

→ Here we will do the opposite.

Best way to solve (*) is to guess form of soln then work out details.

If no collective effects, soln would be

$$[\delta\phi_{\vec k}(\vec J,\omega)]_{\text{bare}} = -(2\pi)^3\sum_{\vec k'}\int d\vec J'\, \frac{\delta f_{\vec k'}(\vec J',0)}{i(\omega-\vec k'\cdot\vec\Omega')}\,\psi_{\vec k\vec k'}(\vec J,\vec J')$$

Ansatz (Rostoker) that full soln is

$$\delta\phi_{\vec k}(\vec J,\omega) = -(2\pi)^3\sum_{\vec k'}\int d\vec J'\, \frac{\delta f_{\vec k'}(\vec J',0)}{i(\omega-\vec k'\cdot\vec\Omega')}\,\psi^d_{\vec k\vec k'}(\vec J,\vec J',\omega) \qquad (\dagger)$$

for some <u>frequency-dependent</u> "dressed interaction" ψ^d.

(Plasma $\psi \to \psi^d$ is Debye screening, $\frac{q}{r} \to \frac{q}{r}e^{-r/\lambda_D}$ in the simplest case.)

Plug this in to (*):

shorthand $\mathrm{Tr}' = \sum_{\vec k'}(2\pi)^3\int d\vec J'$

$$\mathrm{Tr}'\left[\frac{\delta f_{k'}(0)}{i(\omega-k'\cdot\Omega')}\psi^d_{kk'}\right]$$

$$= \mathrm{Tr}'\,\mathrm{Tr}''\left[\frac{\delta f_{k'}(0)}{i(\omega-k'\cdot\Omega')}\psi^d_{k''k'}\,\psi_{kk''}\,\frac{k''\cdot\partial f_0/\partial J''}{\omega-k''\cdot\Omega''}\right]$$

$$+\ \mathrm{Tr}'\left[\frac{\delta f_{k'}(0)}{i(\omega-k'\cdot\Omega')}\psi_{kk'}\right]$$

So $(\dagger)$ is a valid soln if

$$\mathrm{Tr}'\,\frac{\delta f_{k'}(0)}{i(\omega-k'\cdot\Omega')}\left[\psi^d_{kk'}-\psi_{kk'}+\mathrm{Tr}''\,\frac{\psi^d_{k''k'}\,\psi_{kk''}\,k''\cdot\partial f_0/\partial J''}{\omega-k''\Omega''}\right]$$

$$\Rightarrow \psi^d_{\vec k\vec k'}(\vec J,\vec J',\omega) = \psi^d_{\vec k\vec k'}(\vec J,\vec J') - (2\pi)^3\sum_{\vec k''}\int d\vec J''\,\frac{\vec k''\cdot\frac{\partial f_0}{\partial\vec J''}\,\psi_{\vec k\vec k''}(\vec J,\vec J'')\,\psi^d_{\vec k''\vec k'}(\vec J'',\vec J')}{\omega-\vec k''\cdot\vec\Omega''} \qquad (\ddagger)$$

If we can construct ψ^d satisfying $(\ddagger)$, then $(\dagger)$ is soln.

<u>Aside</u>: in plasma none of this gymnastics was necessary.

- $\psi_{\vec k\vec k''} \sim \frac{1}{k^2}\delta^{\vec k''}_{\vec k}$ so no $\sum_{k''}$ on RHS
- solve to get $\psi^d_{\vec k} \sim \frac{\psi_{\vec k}}{\epsilon_{\vec k}}$
- Vlasov + Poisson were diagonalized in same basis, $e^{i\vec k\cdot\vec x}$. Here, Vlasov diag in $e^{i\vec n\cdot\vec\theta}$, but not Poisson! (Because $\delta\phi = \delta\phi(\vec\theta,\vec J)\ldots$)

How to solve $(\ddagger)$? It's not much fun...

<u>Basis method</u>

Assume $\exists$ a set of biorthogonal basis functions $\{\phi^{(p)}(\vec r), \rho^{(p)}(\vec r)\}$ such that

- $\phi^{(p)}(\vec r) = \int d\vec r'\,\psi(\vec r,\vec r')\,\rho^{(p)}(\vec r')$ (self-consistency) (A)
- $\int d\vec r\,[\phi^{(p)}(\vec r)]^*\rho^{(q)}(\vec r) = -\,\mathcal{E}\,\delta^q_p$ (biorthogonality) (B)

arbitrary, unit of energy.

Then any fn. $\chi(\vec r)$ can be written as $\sum_p \chi_p \phi^{(p)}(\vec r)$.
The idea is to expand ψ^d in this basis.
Start with something easier: the bare interaction ψ:

$$\psi(\vec r, \vec r') = -\frac{G}{|\vec r - \vec r'|}$$

At fixed $\vec r'$, can always write

$$\psi(\vec r, \vec r') = \sum_p u_p(\vec r')\, \phi^{(p)}(\vec r)$$

$$\Rightarrow \underbrace{\int d\vec r \,[\rho^{(q)}(\vec r)]^* \psi(\vec r, \vec r')}_{\substack{= \\ [\phi^{(q)}(\vec r')]^* \\ \text{by (A)}}} = \sum_p u_p(\vec r') \underbrace{\int d\vec r \,[\rho^{(q)}(\vec r)]^* \phi^{(p)}(\vec r)}_{\substack{= -\varepsilon \delta_p^q \\ \text{by (B)}}}$$

$$\Rightarrow [\phi^{(q)}(\vec r')]^* = -u_q(\vec r')\,\varepsilon$$

$$\Rightarrow u_q(\vec r') = -\frac{1}{\varepsilon}[\phi^{(q)}(\vec r')]^*$$

$$\Rightarrow \psi(\vec r, \vec r') = -\frac{1}{\varepsilon}\sum_p \phi^{(p)}(\vec r)[\phi^{(q)}(\vec r')]^*$$

$$\Rightarrow \psi_{\vec k \vec k'}(\vec J, \vec J') = -\frac{1}{\varepsilon}\sum_p \phi^{(p)}_{\vec k}(\vec J)[\phi^{(q)}_{\vec k'}(\vec J')]^*.$$

Inspired by this, let's assume ψ^d has the form

$$\psi^d_{\vec k \vec k'}(\vec J, \vec J', \omega) = -\frac{1}{\varepsilon}\sum_{pq} \phi^{(p)}_{\vec k}(\vec J)\, E^{-1}_{pq}(\omega)\,[\phi^{(q)}_{\vec k'}(\vec J')]^*$$

where E is some (as-yet-unknown) dimensionless matrix.
Plugging this into (*), get

$$-\frac{1}{\varepsilon}\sum_p \phi^{(p)}_{\vec k}(\vec J)\, E^{-1}_{pq}(\omega)[\phi^{(q)}_{\vec k'}(\vec J')]^*$$

$$= -\frac{1}{\varepsilon}\sum_{pq} \phi^{(p)}_{\vec k}(\vec J)\,\delta_p^q\,[\phi^{(q)}_{\vec k'}(\vec J')]^*$$

$$-\frac{1}{\varepsilon}\sum_{pq} \phi^{(p)}_{\vec k}(\vec J)\,[M(\omega)E^{-1}(\omega)]_{pq}[\phi_{\vec k'}(\vec J')]^*$$

where

$$M_{pq}(\omega) = \frac{(2\pi)^3}{\varepsilon}\sum_{\vec k}\int d\vec J \,\frac{\vec k \cdot \frac{\partial f_0}{\partial \vec J}}{\omega - \vec k \cdot \vec\Omega}\,\phi^{(p)}_{\vec k}(\vec J)[\phi^{(q)}_{\vec k'}(\vec J')]^*$$

is the "response matrix".

Thus read off: $E^{-1} = \mathbb{1} + M \cdot E^{-1}$

$$\Rightarrow \mathbb{1} = E + M$$

$$\Rightarrow E(\omega) = \mathbb{1} - M(\omega)$$

So that

$$\psi^d_{\vec k \vec k'}(\vec J, \vec J', \omega) = -\frac{1}{\varepsilon}\sum_p \phi^{(p)}_{\vec k}(\vec J)[\mathbb{1} - M(\omega)]_{pq}[\phi^{(q)}_{\vec k'}(\vec J')]^*.$$

$$\left(\begin{array}{l}\underline{\text{Aside}}: \ \det(\mathbb{1}-M(\omega))=0 \text{ gives Landau mode freq. } \{\omega_g\} \\ \quad \text{Reconstruct eigenvector } \vec{X}_g \text{ using } (\mathbb{1}-M(\omega_g))\cdot\vec{X}_g = 0 \\ \quad \Rightarrow \delta\phi_g(\vec{r},t) \propto \sum_p X_g^p \, \phi^{(p)}(\vec{r})\, e^{i\omega_g t}\end{array}\right)$$

If all Landau modes are damped ($\mathrm{Im}\, w_g < 0$) then after sufficient time the ILT of (+) gives the dressed potential fluctuation:

$$\boxed{\delta\phi_{\vec{k}}(\vec{J},t) = (2\pi)^3 \sum_{\vec{k}'} \int d\vec{J}'\, \delta f_{\vec{k}'}(\vec{J}',0)\, e^{-i\vec{k}'\cdot\vec{\Omega}'t}\, \psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')}$$

Rostoker!

- Some examples: spiral eigenmodes,
  radial orbit instability,
  ...

End Monday morning ($t = 6$ hours)

6. Secular evolution

- $$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \vec{J}} \cdot \sum_{\vec{k}} i\vec{k}\, \delta f_{\vec{k}}(\vec{J},t)\, \delta\phi^*_{\vec{k}}(\vec{J},t) \qquad (1)$$

- $$\delta f_{\vec{k}}(\vec{J},t) = \delta f_{\vec{k}}(\vec{J},0)\, e^{-i\vec{k}\cdot\vec{\Omega}t} + i\vec{k}\cdot\frac{\partial f_0}{\partial \vec{J}} \int_0^t dt'\, e^{-i\vec{k}\cdot\vec{\Omega}(t-t')}\, \delta\phi_{\vec{k}}(\vec{J},t') \qquad (2)$$

- $$\delta\phi_{\vec{k}}(\vec{J},t) = (2\pi)^3 \sum_{\vec{k}'} \int d\vec{J}'\, \psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')\, \delta f_{\vec{k}'}(\vec{J}',0)\, e^{-i\vec{k}'\cdot\vec{\Omega}'t} \qquad (3)$$

Ensemble average $\langle .. \rangle$ (many copies of 1 system, or same system time-avg)

$$f_0 = \langle f_0 \rangle$$

$$\Rightarrow \frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \vec{J}} \cdot \sum_{\vec{k}} i\vec{k}\, \langle \delta f_{\vec{k}}(\vec{J},t)\, \delta\phi^*_{\vec{k}}(\vec{J},t) \rangle$$

$$= -\frac{\partial}{\partial \vec{J}} \cdot (\vec{F}_1 + \vec{F}_2)$$

Where

$$\vec{F}_1(\vec{J},t) = \sum_{\vec{k}} i\vec{k}\, e^{-i\vec{k}\cdot\vec{\Omega}t} \langle \delta f_{\vec{k}}(\vec{J},0)\, \delta\phi^*_{\vec{k}}(\vec{J},t) \rangle$$

$$\vec{F}_2(\vec{J},t) = -\sum_{\vec{k}} \vec{k}\vec{k}\cdot\frac{\partial f_0}{\partial \vec{J}} \int_0^t dt'\, e^{i\vec{k}\cdot\vec{\Omega}(t-t')} \langle \delta\phi^*_{\vec{k}}(\vec{J},t)\, \delta\phi_{\vec{k}}(\vec{J},t') \rangle$$

[Physical discussion of friction + diffusion].

Diffusion flux

$$\vec{F}_2(\vec{J},t) = -\hat{D}(\vec{J},t)\cdot\frac{\partial f_0}{\partial \vec{J}} \qquad \hat{D} \text{ is } d\times d \text{ matrix}$$

where

$$\hat{D}(\vec{J},t) = -\sum_{\vec{k}} \vec{k}\vec{k}\int_0^t dt'\, e^{-i\vec{k}\cdot\vec{\Omega}(t-t')}\langle \delta\phi^*_{\vec{k}}(\vec{J},t)\,\delta\phi_{\vec{k}}(\vec{J},t')\rangle$$

Use (3):

$$\langle \delta\phi^*_{\vec{k}}(\vec{J},t)\,\delta\phi_{\vec{k}}(\vec{J},t')\rangle$$

$$= (2\pi)^6 \sum_{\vec{k}'\vec{k}''}\int d\vec{J}'\,d\vec{J}''\, e^{-i(\vec{k}'\cdot\vec{\Omega}'t' - \vec{k}''\cdot\vec{\Omega}''t)}\psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')$$

$$\times [\psi^d_{\vec{k}\vec{k}''}(\vec{J},\vec{J}'',\vec{k}''\cdot\vec{\Omega}'')]^* \underbrace{\langle \delta f_{\vec{k}'}(\vec{J}',0)\,\delta f^*_{\vec{k}''}(\vec{J}'',0)\rangle}_{= m(2\pi)^{-3}\delta^{\vec{k}''}_{\vec{k}'}\delta(\vec{J}'-\vec{J}'')f_0(\vec{J}'),\ \text{see below}}$$

$$= (2\pi)^3 \sum_{\vec{k}'}\int d\vec{J}'\, e^{i\vec{k}'\cdot\vec{\Omega}'(t-t')}|\psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')|^2 f_0(\vec{J}')\,m$$

$$\Rightarrow \hat{D}(\vec{J},t) = -(2\pi)^3 m\sum_{\vec{k}\vec{k}'}\vec{k}\vec{k}$$

$$\times \int d\vec{J}'\, e^{-i(\vec{k}\cdot\vec{\Omega}-\vec{k}'\cdot\vec{\Omega}')t}\,|\psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')|^2 f_0(\vec{J}')$$

$$\times \int_0^t dt'\, e^{i(\vec{k}\cdot\vec{\Omega}-\vec{k}'\cdot\vec{\Omega}')t'}$$

$$= -(2\pi)^3 m\sum_{\vec{k}\vec{k}'}\vec{k}\vec{k}\int d\vec{J}'\left[\frac{1-e^{-i(\vec{k}\cdot\vec{\Omega}-\vec{k}'\cdot\vec{\Omega}')t}}{i(\vec{k}\cdot\vec{\Omega}-\vec{k}'\cdot\vec{\Omega}')}\right]$$

$$\times |\psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')|^2 f_0(\vec{J}')$$

$$\xrightarrow[t \gg \frac{1}{|\vec{k}\cdot\vec{\Omega}|}]{} -\pi(2\pi)^3 m\sum_{\vec{k}\vec{k}'}\int d\vec{J}'\,\delta(\vec{k}\cdot\vec{\Omega}-\vec{k}'\cdot\vec{\Omega}')\,|\psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')|^2 f_0(\vec{J}')$$

Similarly one can show $\vec{F}_1 = \vec{A}f_0$, where

$$\vec{A} = \pi(2\pi)^3 m\sum_{\vec{k}\vec{k}'}\vec{k}\int d\vec{J}'\,\delta(\vec{k}\cdot\vec{\Omega}-\vec{k}'\cdot\vec{\Omega}')\,|\psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')|^2 \times \vec{k}'\cdot\frac{\partial f_0}{\partial \vec{J}'}$$

Then

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \vec{J}}\cdot\left[\vec{A}f_0 - D\cdot\frac{\partial f_0}{\partial \vec{J}}\right] \qquad \text{(Fokker-Planck)}$$

i.e.

$$\frac{\partial f_0}{\partial t} = \pi(2\pi)^3 m\frac{\partial}{\partial \vec{J}}\cdot\sum_{\vec{k}}\vec{k}\int d\vec{J}'\,\delta(\vec{k}\cdot\vec{\Omega}-\vec{k}'\cdot\vec{\Omega}')\,|\psi^d_{\vec{k}\vec{k}'}(\vec{J},\vec{J}',\vec{k}'\cdot\vec{\Omega}')|^2 \times\left(\vec{k}\cdot\frac{\partial}{\partial \vec{J}} - \vec{k}'\cdot\frac{\partial}{\partial \vec{J}'}\right)f_0(\vec{J})\,f_0(\vec{J}')$$

"Balescu-Lenard equation"

- implemented for sphere by Weinberg (1980s), Hamilton+ (2018), Fouvry+ (2021).
- implemented for disk by Fouvry (2015).
- plasma: BL → Landau

$$\int d\vec{v}'\,d\vec{k}'\cdot\frac{g_{\vec{k}\vec{k}'}(\vec{v},\vec{v}')}{|\epsilon_{\vec{k}}(\vec{k}\cdot\vec{v})|^2} \rightarrow \int d\vec{v}'\int_{k_D}^{k_{max}} dk\, g$$

Poisson-noise correlator. $\quad w = (\vec{\theta}, \vec{J})$

$$f^M = f_0 + \delta f$$

$$f^M = m \sum_{i=1}^{N} \delta(w - w_i)$$

$$f_0 = \langle f^M \rangle = f_0(\vec{J})$$

$$\Rightarrow \langle \delta f(w)\, \delta f(w') \rangle = \langle \big(f^M(w) - f_0(w)\big)\big(f^M(w') - f_0(w')\big) \rangle$$

$$= \langle f^M(w) f^M(w') \rangle + f_0(w) f_0(w')$$

$$- \underbrace{\langle f_0(w) f^M(w') \rangle}_{= \langle f_0(w) f_0(w') \rangle} - \underbrace{\langle f_0(w') f^M(w) \rangle}_{=}$$

$$= \langle f^M(w) f^M(w') \rangle - \langle f_0(w) f_0(w') \rangle .$$

Now,

$$\langle f^M(w) f^M(w') \rangle = m^2 \Big\langle \sum_{i=1}^{N} \delta(w - w_i) \sum_{j=1}^{N} \delta(w' - w_j) \Big\rangle$$

$$= m^2 \Big\langle \delta(w - w') \sum_{i=1}^{N} \delta(w - w_i) + \sum_{i=1}^{N} \sum_{j \neq i} \delta(w - w_i)\, \delta(w' - w_j) \Big\rangle$$

assume uncorrelated

$$= m\, \delta(w - w') \Big\langle m \sum_{i=1}^{N} \delta(w - w_i) \Big\rangle + m^2 \sum_{i=1}^{N} \sum_{j \neq i} \langle \delta(w - w_i) \rangle \langle \delta(w' - w_j) \rangle$$

$$= m\, \delta(w - w') \overbrace{\langle f^M(w) \rangle}^{= f_0} + \underbrace{\frac{N(N-1)}{N^2}}_{= 1 - \frac{1}{N}} f_0(\vec{J}) f_0(\vec{J}')$$

$$= m\, \delta(w - w') f_0(\vec{J}) + \left(1 - \frac{1}{N}\right) f_0(\vec{J}) f_0(\vec{J}')$$

Hence,

$$\langle \delta f(w)\, \delta f(w') \rangle = m\, \delta(w - w') f_0(\vec{J}) + \left(\cancel{1} - \frac{1}{N}\right) f_0(\vec{J}) f_0(\vec{J}') \cancel{- f_0(\vec{J}) f_0(\vec{J}')}$$

$$= m\, \delta(w - w') f_0(\vec{J}) - \frac{1}{N} f_0(\vec{J}) f_0(\vec{J}')$$

$$\Rightarrow \langle \delta f(\vec{\theta}, \vec{J}, 0)\, \delta f(\vec{\theta}', \vec{J}', 0) \rangle = m\, \delta(\vec{\theta} - \vec{\theta}')\, \delta(\vec{J} - \vec{J}') f_0(\vec{J}) - \frac{1}{N} f_0(\vec{J}) f_0(\vec{J}')$$

$$\Rightarrow \langle \delta f_{\vec{k}}(\vec{J}, 0)\, \delta f^*_{\vec{k}'}(\vec{J}', 0) \rangle = \frac{m}{(2\pi)^3} \delta_{\vec{k}}^{\vec{k}'}\, \delta(\vec{J} - \vec{J}') f_0(\vec{J}) \underbrace{- \frac{1}{N} \delta_{\vec{k}}^{\vec{0}}\, \delta_{\vec{k}'}^{\vec{0}} f_0(\vec{J}) f_0(\vec{J}')}_{\text{always contributes zero.}}$$