Kinetic Theory of Stellar Systems
Problem Set

Chris Hamilton

Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540
Email: chamilton@ias.edu

Last updated: 3 October 2024

1. Basic notions of angle-action variables.

(a) Consider a test particle moving in a Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p})$ where $\boldsymbol{q}$ and $\boldsymbol{p}$ are canonically conjugate coordinates and momenta. Write down Hamilton's equations for $\boldsymbol{q}$ and $\boldsymbol{p}$.

(b) Explain why orbits in smooth, inhomogeneous gravitational potentials are naturally described using angle-action variables $(\boldsymbol{\theta}, \boldsymbol{J})$ rather than position-velocity $(\boldsymbol{r}, \boldsymbol{v})$. Use angle-action coordinates to write down expressions for the Poisson bracket $[g, h]$ and the phase space volume element $\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}$.

(c) Supposing $H = H_0(\boldsymbol{J})$, write down an expression for the frequency $\boldsymbol{\Omega}(\boldsymbol{J})$. In a static, non-rotating, spherically symmetric potential the natural action coordinates are $\boldsymbol{J} = (J_r, L, L_z)$, where J_r is the particle's radial action, L is the norm of its specific angular momentum, and L_z is the z-component of that angular momentum. In this case one can always write $H_0(J_r, L)$. What is Ω_3 in this case? What is the physical significance of this result? Sketch two orbits in a generic spherically symmetric potential (but not the Keplerian or harmonic potentials), one with $J_r \gg L$ and one with $J_r \ll L$.

2. Bar-halo friction.

This question is taken from the 2023 exam.

(a) Consider two arbitrary smooth functions of angle-action coordinates, $g(\boldsymbol{\theta}, \boldsymbol{J})$ and $h(\boldsymbol{\theta}, \boldsymbol{J})$. Using these coordinates write down the expression for the Poisson bracket $[g, h]$.

(b) Let f be the distribution function of an ensemble of particles whose motion is governed by the 'mean field + perturbation' Hamiltonian

$$H(\boldsymbol{\theta}, \boldsymbol{J}, t) = H_0(\boldsymbol{J}) + \delta\Phi(\boldsymbol{\theta}, \boldsymbol{J}, t). \tag{2.1}$$

The equation governing the evolution of f is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + [f, H] = 0. \tag{2.2}$$

Let $f(\boldsymbol{\theta}, \boldsymbol{J}, t) = f_0(\boldsymbol{J}) + \delta f(\boldsymbol{\theta}, \boldsymbol{J}, t)$, where $f_0(\boldsymbol{J})$ is the unperturbed DF. Fourier expanding the potential as $\delta\Phi = \sum_{\mathbf{k}} \delta\Phi_{\mathbf{k}}(\boldsymbol{J}, t) \exp(i\mathbf{k} \cdot \boldsymbol{\theta})$ and similarly for δf, where $\boldsymbol{k} \in \mathbb{Z}^3$, assuming all perturbations are small, and assuming that $\delta\Phi$ is switched on at $t = 0$ and that $\delta f_{\boldsymbol{k}}(\boldsymbol{J},0)=0$, show that the linear response of the DF satisfies

$$\delta f_{\mathbf{k}}(\boldsymbol{J},t)=i\mathbf{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{J}}\int_0^t \mathrm{d}t'\delta\Phi_{\mathbf{k}}(\boldsymbol{J},t')\mathrm{e}^{-i\mathbf{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})(t-t')}, \tag{2.3}$$

where you should define the frequency vector $\boldsymbol{\Omega}(\boldsymbol{J})$.

(c) Consider a galaxy consisting of an initially spherically symmetric dark matter halo, plus a rigidly rotating 'bar' of stars which is centred at the origin and which rotates anticlockwise about the z axis. In the absence of the bar, the dynamics of a dark matter particle is governed by the Hamiltonian $H_0(\boldsymbol{J})$. Let the bar perturbation have potential $\delta\Phi$. Each dark matter particle then moves in the time-dependent Hamiltonian $H=H_0+\delta\Phi$. Let the dark matter distribution function be f, normalized such that $\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{J} f=1$. Suitable angle-action coordinates for describing this system are

$$\boldsymbol{\theta}=(\theta_r,\theta_\psi,\theta_\varphi), \qquad \boldsymbol{J}=(J_r,L,L_z), \tag{2.4}$$

where in particular, J_r is the radial action, L is the specific angular momentum, and L_z is the z-component of specific angular momentum. Write down Hamilton's equation for the evolution of L_z. Hence show that the total torque induced by the bar upon the halo, per unit halo mass, is equal to

$$\mathcal{T}(t)=-\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{J}\, f(\boldsymbol{\theta},\boldsymbol{J},t)\frac{\partial\delta\Phi(\boldsymbol{\theta},\boldsymbol{J},t)}{\partial\theta_\varphi}. \tag{2.5}$$

Making the same expansions as in part (a), show that

$$\mathcal{T}(t)=\sum_{\mathbf{k}} i(2\pi)^3 k_\varphi \int \mathrm{d}\boldsymbol{J}\,\delta f_{\boldsymbol{k}}(\boldsymbol{J},t)\delta\Phi^*_{\boldsymbol{k}}(\boldsymbol{J},t), \tag{2.6}$$

where $\boldsymbol{k}=(k_r,k_\psi,k_\varphi)$ are integer vectors. You may quote the identity $\int \mathrm{d}\boldsymbol{\theta}\exp(i\boldsymbol{k}\cdot\boldsymbol{\theta})=(2\pi)^3\delta_{\mathbf{k}}^{\mathbf{0}}$.

(d) Let the bar rotate anticlockwise with fixed angular speed $\Omega_\mathrm{p}>0$; then we can write $\delta\Phi_{\mathbf{k}}(\boldsymbol{J},t)=\Psi_{\mathbf{k}}(\boldsymbol{J})\exp(-ik_\varphi\Omega_\mathrm{p}t)$. By combining this with equations (2.3) and (2.6), and stating any further assumptions you make, show that in the limit $t\to\infty$, the torque on the dark matter halo reaches a steady-state value:

$$\mathcal{T}=-\sum_{\boldsymbol{k}}\pi(2\pi)^3k_\varphi\int \mathrm{d}\boldsymbol{J}|\Psi_{\mathbf{k}}|^2\mathbf{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{J}}\delta\left(\mathbf{k}\cdot\boldsymbol{\Omega}-k_\varphi\Omega_\mathrm{p}\right). \tag{2.7}$$

(e) Suppose $f_0(\boldsymbol{J})$ depends on $\boldsymbol{J}$ only through the mean field energy, i.e. $f_0=f_0(E)$ where $E=H_0(\boldsymbol{J})$, and that $\mathrm{d}f_0/\mathrm{d}E<0$. Show that in this case $\mathcal{T}$ is positive definite. What might this result imply about the long-term evolution of stellar bars? (You may assume the bar has a positive moment of inertia.)

3. Linear response in homogeneous stellar systems.

The dimensionless response matrix of a stellar system is given by

$$\mathsf{M}_{pq}(\omega)=\frac{(2\pi)^3}{\mathcal{E}}\sum_{\boldsymbol{k}}\int \mathrm{d}\boldsymbol{J}\,\frac{\boldsymbol{k}\cdot\partial f_0/\partial\boldsymbol{J}}{\omega-\boldsymbol{k}\cdot\boldsymbol{\Omega}}[\Phi_{\boldsymbol{k}}^{(p)}(\boldsymbol{J})]^*\Phi_{\boldsymbol{k}}^{(q)}(\boldsymbol{J}) \tag{3.1}$$

In this exercise, we set out to show that for homogeneous systems this response matrix reduces to the dielectric function of plasma physics.

(a) Assume that the system is placed within a periodic 3D box of size L, and that the mean potential vanishes, i.e. $\Phi_0 = 0$, so that unperturbed trajectories are straight lines. The system's angle-action coordinates and the associated orbital frequencies are given by

$$\boldsymbol{\theta} = \frac{2\pi}{L}\boldsymbol{r}, \qquad \boldsymbol{J} = \frac{L}{2\pi}\boldsymbol{v} \qquad \boldsymbol{\Omega} = \frac{2\pi}{L}\boldsymbol{v}. \tag{3.2}$$

The system's instantaneous potential and DF are linked by

$$\Phi(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{r}'\mathrm{d}\boldsymbol{v}' f(\boldsymbol{r}', \boldsymbol{v}')\psi(\boldsymbol{r}, \boldsymbol{r}'), \tag{3.3}$$

where $\psi(\boldsymbol{r}, \boldsymbol{r}') = -G/|\boldsymbol{r} - \boldsymbol{r}'|$ is the gravitational pairwise interaction. Making use of the periodicity of the system and assuming $\psi(\boldsymbol{\theta}, \boldsymbol{\theta}')$ is translationally invariant, show that

$$\psi(\boldsymbol{\theta}, \boldsymbol{\theta}) = -\frac{G}{L\pi}\sum_{\boldsymbol{p}\neq\boldsymbol{0}} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{\theta}-\boldsymbol{\theta}')}}{|\boldsymbol{p}|^2} \tag{3.4}$$

Using this, show that the response matrix becomes

$$\mathsf{M}_{\boldsymbol{pq}}(\omega) = \frac{GL^2}{\pi}\delta_{\boldsymbol{p}}^{\boldsymbol{q}}\frac{1}{|\boldsymbol{p}|^2}\int \mathrm{d}\boldsymbol{v}\, \frac{\boldsymbol{p}\cdot\partial f_0/\partial\boldsymbol{v}}{\overline{\omega} - \boldsymbol{p}\cdot\boldsymbol{v}}, \tag{3.5}$$

where $\overline{\omega} \equiv (2\pi)^{-1}L\omega$.

(b) Assume that the system's mean distribution function (DF) follows the Maxwellian distribution:

$$f_0(\boldsymbol{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}}\mathrm{e}^{-|\boldsymbol{v}|^2/(2\sigma^2)}, \tag{3.6}$$

where ρ_0 is the system's mean density, and σ is the velocity dispersion. Show that in this case

$$\mathsf{M}_{\boldsymbol{pq}}(\omega) = \delta_{\boldsymbol{p}}^{\boldsymbol{q}}\frac{1}{|\boldsymbol{p}|^2}\left(\frac{L}{L_\mathrm{J}}\right)^2[1 + \zeta Z(\zeta)], \tag{3.7}$$

where

$$\zeta = \frac{\overline{\omega}}{\sqrt{2}|\boldsymbol{p}|\sigma}, \qquad Z(\zeta) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}u\frac{\mathrm{e}^{-u^2}}{u-\zeta}, \qquad L_\mathrm{J} = \sqrt{\frac{\pi\sigma^2}{G\rho_0}}. \tag{3.8}$$

What is the main difference between this expression for a self-gravitating system, and the analogous one for an electrostatic plasma? What happens for a system with $L > L_\mathrm{J}$? (Hint: write down the dispersion relation and then look for purely growing solutions $\omega = i\gamma$ for real γ).

4. Using Rostoker's principle to derive the Balescu-Lenard equation.

Rostoker's principle tells us that we can think of a secularly evolving plasma or stellar system as a collection of N uncorrelated particles undergoing two-body encounters (a la Chandrasekhar), provided we replace the bare Coulombic/Newtonian interaction $\psi(\boldsymbol{r}, \boldsymbol{r}')$ of each two-body encounter with its dressed counterpart ψ^d. In this exercise we use this principle to derive the BL equation in a short and simple way. Note we do not need to deal explicitly with fluctuations here, so we drop the '0' subscript on mean-field quantities.

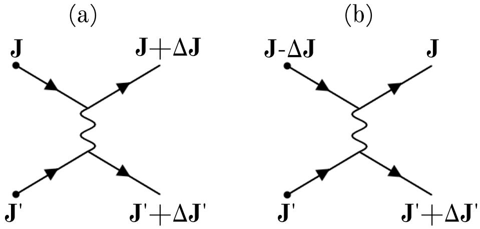

FIGURE 1. Two possibilities for 'scattering' in action coordinates. A test star's action changes by $\Delta\boldsymbol{J}$ during an 'encounter' with a field star whose action changes from $\boldsymbol{J}'$ to $\boldsymbol{J}' + \Delta\boldsymbol{J}'$. The DF $f(\boldsymbol{J})$ will be decremented if the test star is kicked out of state $\boldsymbol{J}$ (process (a)), but will be incremented if it is kicked in to state $\boldsymbol{J}$ from $\boldsymbol{J} - \Delta\boldsymbol{J}$ (process (b)). To write down a master equation we must also account for the inverse processes (by reversing the directions of the arrows).

4.1. *Interaction of two stars via a dressed potential*

Consider a 'test' star with coordinates $(\boldsymbol{\theta}, \boldsymbol{J})$ and a 'field' star with coordinates $(\boldsymbol{\theta}', \boldsymbol{J}')$. Rostoker's principle says that we can forget about all the other stars, and treat the interaction of these two stars as if they were an isolated system with specific two-body Hamiltonian (units of (velocity)^2):

$$h = H(\boldsymbol{J}) + H(\boldsymbol{J}') + U^{\mathrm{d}}(\boldsymbol{\theta}, \boldsymbol{J}, \boldsymbol{\theta}', \boldsymbol{J}'), \tag{4.1}$$

where $H(\boldsymbol{J})$ is the mean-field Hamiltonian. Here $U^{\mathrm{d}}(\boldsymbol{\theta}, \boldsymbol{J}, \boldsymbol{\theta}', \boldsymbol{J}')$ is the dressed specific potential energy between a particle at phase space location $(\boldsymbol{\theta}, \boldsymbol{J})$ and a particle at $(\boldsymbol{\theta}', \boldsymbol{J}')$. (It consists of the usual Newtonian attraction plus collective effects; if we ignore these then $U^{\mathrm{d}} \to -Gm/|\boldsymbol{r} - \boldsymbol{r}'|$). Let us expand ψ^{d} as a Fourier series in the angle variables:

$$\psi^{\mathrm{d}}(\boldsymbol{\theta}, \boldsymbol{J}, \boldsymbol{\theta}', \boldsymbol{J}') = m \sum_{\boldsymbol{k}\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\theta} - \boldsymbol{k}'\cdot\boldsymbol{\theta}')}\, \psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{k}'\cdot\boldsymbol{\Omega}'). \tag{4.2}$$

Here $\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{k}'\cdot\boldsymbol{\Omega}')$ is the dressed potential interaction we derived in the lectures, and we have put in the correct frequency dependence ($\omega = \boldsymbol{k}'\cdot\boldsymbol{\Omega}'$), basically because we know what the answer has to look like, but also because it is physically clear that these are the only frequencies available to the system.

Treat the two-body interaction as a perturbation. To zeroth order in this perturbation the test and field stars just follow their mean field trajectories $\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \boldsymbol{\Omega}t$ and $\boldsymbol{\theta}' = \boldsymbol{\theta}'_0 + \boldsymbol{\Omega}'t$ indefinitely. To first order, the result of their interaction is to nudge each other to new actions $\boldsymbol{J} + \delta\boldsymbol{J}$ and $\boldsymbol{J}' + \delta\boldsymbol{J}'$ respectively. Show that

$$\delta\boldsymbol{J}(\boldsymbol{\theta}_0, \boldsymbol{J}, \boldsymbol{\theta}'_0, \boldsymbol{J}', \tau) = -m \sum_{\boldsymbol{k}\boldsymbol{k}'} \mathrm{i}\boldsymbol{k}\, \psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{k}'\cdot\boldsymbol{\Omega}')\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\theta}_0 - \boldsymbol{k}'\cdot\boldsymbol{\theta}'_0)} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')\tau} - 1}{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')}, \tag{4.3}$$

The result for $\delta\boldsymbol{J}'$ is the same as (4.3) except we replace the first factor $\mathrm{i}\boldsymbol{k} \to -\mathrm{i}\boldsymbol{k}'$.

4.2. *The master equation*

Following Rostoker, we consider the relaxation of our entire system to consist of nothing more than an uncorrelated set of dressed two-body encounters. Then it is easy to write down a master equation for the DF $f(\boldsymbol{J})$. To do so, we account for (a) test stars being nudged out of the state $\boldsymbol{J}$ and in to some new state $\boldsymbol{J} + \Delta\boldsymbol{J}$, as illustrated in Figure 1a, and (b) test stars being nudged in to the state $\boldsymbol{J}$ from $\boldsymbol{J} - \Delta\boldsymbol{J}$, as in Figure 1b. Processes

(a) and (b) are both deterministic and time-reversible, so we must also account for their inverses, which would be represented by the same diagrams but with the arrows pointing in the opposite direction.

Let the *transition rate density* be $w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}, \boldsymbol{J}')$. This quantity is defined such that $w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}, \boldsymbol{J}')\,\mathrm{d}\Delta\boldsymbol{J}\mathrm{d}\Delta\boldsymbol{J}'\tau$ is the probability that a given test star with initial action $\boldsymbol{J}$ is scattered to the volume of phase space $\mathrm{d}\Delta\boldsymbol{J}$ around $\boldsymbol{J}+\Delta\boldsymbol{J}$, by a given field star with action $\boldsymbol{J}'$ that is itself scattered to the volume element $\mathrm{d}\Delta\boldsymbol{J}'$ around $\boldsymbol{J}'+\Delta\boldsymbol{J}'$, in a time interval τ that is much longer than an orbital period but much shorter than the relaxation time. Assuming the system's equilibrium state is invariant under time reversal, argue that f satisfies the master equation

$$\begin{aligned}\frac{\partial f(\boldsymbol{J})}{\partial t} &= \frac{(2\pi)^3}{m}\int \mathrm{d}\boldsymbol{J}'\mathrm{d}\Delta\boldsymbol{J}\mathrm{d}\Delta\boldsymbol{J}' \\ &\times \frac{1}{2}\Big[w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}, \boldsymbol{J}')[-f(\boldsymbol{J})f(\boldsymbol{J}') + f(\boldsymbol{J}+\Delta\boldsymbol{J})f(\boldsymbol{J}'+\Delta\boldsymbol{J}')] \\ &+ w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}-\Delta\boldsymbol{J}, \boldsymbol{J}')[f(\boldsymbol{J}-\Delta\boldsymbol{J})f(\boldsymbol{J}') - f(\boldsymbol{J})f(\boldsymbol{J}'+\Delta\boldsymbol{J}')]\Big]. \qquad (4.4)\end{aligned}$$

By expanding the integrand for weak interactions, i.e. for $\Delta\boldsymbol{J}, \Delta\boldsymbol{J}' \ll \boldsymbol{J}, \boldsymbol{J}'$, up to second order in small quantities, show that

$$\frac{\partial f(\boldsymbol{J})}{\partial t} = \frac{\partial}{\partial \boldsymbol{J}}\cdot\int \mathrm{d}\boldsymbol{J}'\left[\mathsf{A}\cdot f(\boldsymbol{J}')\frac{\partial f}{\partial \boldsymbol{J}} + \mathsf{B}\cdot f(\boldsymbol{J})\frac{\partial f}{\partial \boldsymbol{J}'}\right]. \qquad (4.5)$$

where A is a 3×3 matrix:

$$\mathsf{A}(\boldsymbol{J}, \boldsymbol{J}') = \frac{(2\pi)^3}{2m}\int \mathrm{d}\Delta\boldsymbol{J}\mathrm{d}\Delta\boldsymbol{J}' w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}, \boldsymbol{J}')\,\Delta\boldsymbol{J}\Delta\boldsymbol{J} \equiv \frac{(2\pi)^3}{2m}\frac{\langle\Delta\boldsymbol{J}\Delta\boldsymbol{J}\rangle_\tau}{\tau}, \qquad (4.6)$$

where $\langle\Delta\boldsymbol{J}\Delta\boldsymbol{J}\rangle_\tau$ is the expectation value of $\Delta\boldsymbol{J}\Delta\boldsymbol{J}$ after a time interval τ for a given test star action $\boldsymbol{J}$ and field star action $\boldsymbol{J}'$. What is the analogous expression for B?

4.3. *The kinetic equation*

Now we put the results of §4.1 and §4.2 together. Since by Jeans' theorem stars are uniformly distributed in the angle variables, we can calculate the expectation value $\langle\Delta\boldsymbol{J}\Delta\boldsymbol{J}\rangle_\tau$ by averaging over initial phases $\boldsymbol{\theta}_0, \boldsymbol{\theta}_0'$. Thus we have

$$\langle\Delta\boldsymbol{J}\Delta\boldsymbol{J}\rangle_\tau = \int \frac{\mathrm{d}\boldsymbol{\theta}_0}{(2\pi)^3}\frac{\mathrm{d}\boldsymbol{\theta}_0'}{(2\pi)^3}\,\delta\boldsymbol{J}(\boldsymbol{\theta}_0, \boldsymbol{J}, \boldsymbol{\theta}_0', \boldsymbol{J}', \tau)\,\delta\boldsymbol{J}(\boldsymbol{\theta}_0, \boldsymbol{J}, \boldsymbol{\theta}_0', \boldsymbol{J}', \tau), \qquad (4.7)$$

where $\delta\boldsymbol{J}$ is given in equation (4.3). Plugging (4.7) and (4.3) in to (4.6) and taking the limit $\tau\to\infty$ show that

$$\begin{aligned}\mathsf{A}(\boldsymbol{J}, \boldsymbol{J}') &= \frac{m}{2(2\pi)^3}\int \mathrm{d}\boldsymbol{\theta}_0\mathrm{d}\boldsymbol{\theta}_0'\sum_{\boldsymbol{k}\boldsymbol{k}'}\sum_{\boldsymbol{q}\boldsymbol{q}'}\boldsymbol{k}\,\boldsymbol{q}\,\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{k}'\cdot\boldsymbol{\Omega}')\,\psi^{\mathrm{d}}_{\boldsymbol{q}\boldsymbol{q}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{q}'\cdot\boldsymbol{\Omega}') \\ &\times \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{\theta}_0}\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}'+\boldsymbol{q}')\cdot\boldsymbol{\theta}_0'}\lim_{\tau\to\infty}\left[\tau^{-1}\frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\Omega}-\boldsymbol{k}'\cdot\boldsymbol{\Omega}')\tau}-1}{\boldsymbol{k}\cdot\boldsymbol{\Omega}-\boldsymbol{k}'\cdot\boldsymbol{\Omega}'}\frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{\Omega}-\boldsymbol{q}'\cdot\boldsymbol{\Omega}')\tau}-1}{\boldsymbol{q}\cdot\boldsymbol{\Omega}-\boldsymbol{q}'\cdot\boldsymbol{\Omega}'}\right].\end{aligned} \qquad (4.8)$$

Making use of the following two identities

$$\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}') = [\psi^{\mathrm{d}}_{-\boldsymbol{k},-\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}')]^*, \qquad \text{and} \qquad \lim_{\tau\to\infty}\left[|\mathrm{e}^{\mathrm{i}x\tau}-1|^2/x^2\tau\right] = 2\pi\delta(x), \qquad (4.9)$$

show that

$$\mathsf{A}(\boldsymbol{J},\boldsymbol{J}') = \pi(2\pi)^3 m \sum_{\boldsymbol{k}\boldsymbol{k}'} \boldsymbol{k}\,\boldsymbol{k}\,\delta(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')|\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}'\cdot\boldsymbol{\Omega}')|^2. \tag{4.10}$$

The result for B is identical to (4.10) except we replace the factor $\boldsymbol{k}\boldsymbol{k}$ with $-\boldsymbol{k}\boldsymbol{k}'$. Putting the explicit formulae for $\mathsf{A}(\boldsymbol{J},\boldsymbol{J}')$ and $\mathsf{B}(\boldsymbol{J},\boldsymbol{J}')$ in to the kinetic equation (4.5), recover the Balescu-Lenard equation:

$$\frac{\partial f(\boldsymbol{J})}{\partial t} = \pi(2\pi)^3 m \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k}\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' \delta(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')|\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}'\cdot\boldsymbol{\Omega}')|^2 \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) f(\boldsymbol{J}) f(\boldsymbol{J}'). \tag{4.11}$$

5. Mass, energy and entropy.

Up to prefactors, the total mass, mean field energy and entropy of a stellar system are

$$\mathcal{M}(t) = \int \mathrm{d}\boldsymbol{J}\, f_0(\boldsymbol{J},t), \tag{5.1}$$

$$\mathcal{E}(t) = \int \mathrm{d}\boldsymbol{J}\, f_0(\boldsymbol{J},t)\, H_0(\boldsymbol{J}), \tag{5.2}$$

$$\mathcal{S}(t) = -\int \mathrm{d}\boldsymbol{J}\, f_0(\boldsymbol{J},t) \ln f_0(\boldsymbol{J},t). \tag{5.3}$$

(a) Assume that the secular evolution is instead self-consistently driven by the (collectively amplified) finite-N noise in the system, so that f_0 satisfies the Balescu-Lenard kinetic equation (i.e. equation (4.11) for $f \to f_0$). Show that (i) $\mathrm{d}\mathcal{M}/\mathrm{d}t = 0$, (ii) $\mathrm{d}\mathcal{E}/\mathrm{d}t = 0$ and (iii) $\mathrm{d}\mathcal{S}/\mathrm{d}t \geqslant 0$, and comment upon your results. [You may assume the symmetry $|\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\omega)|^2 = |\psi^{\mathrm{d}}_{\boldsymbol{k}'\boldsymbol{k}}(\boldsymbol{J}',\boldsymbol{J},\omega)|^2$ for real ω].

(b) Suppose there existed a distribution $f_0 \propto \exp(-\beta H_0(\boldsymbol{J}))$ for some constant β with units of (velocity)$^{-2}$. Compute $\partial f_0/\partial t$ as driven by the Balescu-Lenard equation in this case, and comment on your result.